# DETERMINATION OF THE UNKNOWN PARAMETERS OF AN OIL-BEARING STRATUM IN THE PRESENCE OF RETURN FLOWS THROUGH A WEAKLY PERMEABLE STRATUM AND INFILTRATION

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The article discusses the inverse problem for a parabolic equation describing filtration in the presence of return flows through a weakly permeable stratum and infiltration. It discusses a system of redefined boundary conditions, uniquely determining the unknown functions entering into the filtration equation.

In [1] the inverse problem of the theory of filtration was discussed for the simplest model of flow in a pressurized stratum.

The unknown permeability coefficient was determined from the given pressure and the output of a central well. The solution was obtained by methods developed in [2-4].

In the present article, the inverse problem is solved for a more complex filtration model, taking account of return flows through a weakly permeable layer and of infiltration.

Let us consider a circular stratum of radius R and a central well with the radius $r_0$. We shall assume the filtration parameters of the stratum to be radially symmetrical. In the presence of return flows through an underlying layer, the equation for the pressure has the form (the coefficient of the elastic capacity, for simplicity, is taken equal to unity) [5]

$$\frac{\partial p}{\partial t}=\frac{1}{r}\frac{\partial}{\partial r}\left[rk(r)\frac{\partial p}{\partial r}\right]-h(r)p \tag{1}$$

Here k(r) is the permeability coefficient of the main stratum; the function h(r) is inversely proportional to the thickness of the underlying weakly permeable layer.

The inverse problem for Eq. (1) consists in determining the unknowns q(t) and h(r) from certain information on the solution p(r, t).

Let us consider a redefined system of boundary conditions [the mass flow rate q(t) and the pressure $\varphi(t)$ in the well will be assumed to be given functions of the time]:

$$2\pi r_0 k(r_0)\frac{\partial p}{\partial r}\Big|_{r=r_0}=q(t),\quad p_{r=R}=0,\quad p_{t=0}=0 \tag{2}$$

$$p_{r=r_0}=\varphi(t) \tag{3}$$

It was shown in [1] that with $h(r)\equiv 0$ the coefficient k(r) is uniquely established from the functions $\varphi(t)$ and q(t). We shall show that in the present case the condition (2), (3) is sufficient for the single-valued determination of the functions k(r), h(r). Let us solve Eqs. (1), (2). We shall seek the solution in the form of a series in terms of the eigenfunctions of the operator

$$Lp=-\frac{1}{r}\frac{\partial}{\partial r}\left[rk(r)\frac{\partial p}{\partial r}\right]+h(r)p=\lambda p \tag{4}$$

$$p'(r_0)=0,\ p(R)=0$$

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As in [1], after a Laplace transformation with respect to the time, we obtain

$$P(r,s) = -Q(s)\sum_{k=1}^{\infty}\frac{p_k(r_0)}{\lambda_k + s}p_k(r) \tag{5}$$

$$P(r,s) = \int_0^{\infty} p(r,t)e^{-st}dt, \quad Q(s) = \int_0^{\infty} q(t)e^{-st}dt$$

Here $p_k(r)$, $\lambda_k$ are normalized eigenfunctions and eigennumbers of the operator (4).

We shall assume that the functions k(r), h(r) satisfy the conditions

$$k(r) \in C^2[r_0, R], \; k(r) > 0$$
$$h(r) \in C[r_0, R], \; h(r) > M > -\infty$$

Then the operator (4) has a finite number of negative eigennumbers and, by virtue of the Mercer theorem, series (5) converges uniformly to $[r_0, R]$.

Setting $r = r_0$ in (5), we obtain [$\alpha_k = p_k^{-2}(r_0)$, the normalized factors of the operator (4)]:

$$\Phi(s) = -Q(s)\sum_{k=1}^{\infty}\frac{p_k^2(r_0)}{\lambda_k + s} = -Q(s)\sum_{k=1}^{\infty}\frac{1}{\alpha_k(\lambda_k + s)}$$

$$\Phi(s) = \int_0^{\infty}\varphi(t)e^{-st}dt$$

From the last equality it follows that

$$-\frac{\Phi(s)}{Q(s)} = \sum_{k=1}^{\infty}\frac{1}{\alpha_k(\lambda_k + s)} \tag{6}$$

From (6) we can determine the spectral parameters of the operator (4). The eigennumbers $\lambda_k$ are the poles of the expression $\Phi(-\lambda)/Q(-\lambda)$, and the normalized factors are determined by the appropriate deductions:

$$\alpha_k^{-1} = \mathrm{Res}_{\lambda=\lambda_k}[\Phi(-\lambda)/Q(-\lambda)]$$

We make the following replacement of variables in Eq. (4) [6]:

$$x = \frac{\pi}{B}\int_{r_0}^{r}\frac{dr}{\sqrt{k(r)}}, \quad r\sqrt{k(r)} = \theta(x) \tag{7}$$

$$z(x) = p\sqrt{\theta}, \quad B = \int_{r_0}^{R}\frac{dr}{\sqrt{k(r)}}$$

After the replacement, we obtain

$$-z'' + f(x)z = \mu z \tag{8}$$
$$z'(0) - \frac{\theta'(0)}{2\theta(0)}z(0) = 0, \quad z(\pi) = 0$$

$$f(x) = l(x) + \frac{B^2}{\pi^2}h(x), \quad l(x) = \frac{(\sqrt{\theta(x)})''}{\sqrt{\theta(x)}} \tag{9}$$
$$\mu = \frac{B^2}{\pi^2}\lambda$$

We will call operator (8) an operator corresponding to operator (4) and Eq. (1) with conditions (2), (3).

The eigennumbers $\mu_k$ and the normalized factors $\beta_k$ of operator (8) are equal to

$$\mu_k = \frac{B^2}{\pi^2}\lambda_k, \quad \beta_k = \frac{\pi}{B\theta(0)}\alpha_k$$

The constants B and $\theta(0)$ are determined from the asymptotic formulas for $\mu_k$ and $\beta_k$

$$\sqrt{\mu_k} = k + \frac{1}{2} + O\left(\frac{1}{k}\right), \quad \beta_k = \frac{\pi}{2} + O\left(\frac{1}{k}\right)$$
$$B = \pi\lim_{k\to\infty}\frac{k}{\sqrt{\lambda_k}}, \quad \theta(0) = \frac{2}{B}\lim_{k\to\infty}\alpha_k$$

Thus, the system of redefined boundary conditions (2), (3) completely determines the spectral function of the corresponding operator (8), in accordance with which the operator is uniquely established [2]. As in [1], the function $f(x)$ and the constant $\theta'(0)$ are determined from the solution of the inverse Sturm–Liouville problem for the operator (8), with the spectral function [2]

$$\sigma(\mu) = \sum_{\mu \leq \mu_k} \beta_k^{-1}$$

Let us consider the case where one of the functions k(r) or h(r) is known at $[r_0, R]$.

If h(r) is known, for the $y = r^2(x)$, taking account of (7), (9), we can obtain the nonlinear differential equation:

$$\frac{(\sqrt{y'})''}{\sqrt{y'}} + \frac{B^2}{\pi^2} h(\sqrt{y}) = f(x)$$
$$y(0) = r_0^2, \quad y'(0) = \frac{2B}{\pi}\theta(0), \quad y''(0) = \frac{2B}{\pi}\theta'(0)$$

The function k(r) is determined in parametric form:

$$r(x) = \sqrt{y(x)}, \quad k(x) = \frac{\pi^2}{4B^2}\frac{y'^2(x)}{y(x)} \tag{10}$$

If the function k(r) is known, then, consequently, the functions r(x), $\theta(x)$, $l(x)$ are known. Then, going over from the variable x to r, we find the function h(r):

$$h(r) = \frac{\pi^2}{B^2}[f(r) - l(r)], \quad x(r) = \frac{\pi}{B} I(r), \quad I(r) = \int_{r_0}^{r} \frac{dr}{\sqrt{k(r)}} \tag{11}$$

If both functions are unknown, the supplementary condition (3) is obviously insufficient for the single-valued determination of k(r) and h(r). There naturally arises the question of what additional information with respect to the functions k(r) and h(r) can be obtained by assigning values of the solution p(r, t) not at the single point $r = r_0$, but at several points $r_m$, m = 1, 2, . . ., N.

It is found that no number of additional functions $p(r_m, t)$ will permit the single-valued determination of k(r) and h(r).

Before proving this, let us consider the following problem.

Let there be given the redefined system (1)-(3), the value of the function $k(r_1)$ at the point $r = r_1$, and the integral $x_1(r_1) = \pi B^{-1} I(r_1)$.

We shall show that in this case the solution of the direct problem $p(r_1, t)$ at the point $r = r_1$ can be determined for all moments of time.

In actuality, the corresponding operator (8) is determined by conditions (2), (3). Consequently, the normalized eigenfunctions of the operator (8), $z_k(x)$ are known. But then the following values are known:

$$p_k(r_1) = \sqrt{\frac{\pi}{B}}\frac{z_k(x_1)}{r_1^{1/2} k^{1/4}(r_1)} \tag{12}$$

Substituting (12) into the series (5) we obtain

$$P(r_1, s) = -Q(s)\sum_{k=1}^{\infty}\sqrt{\frac{\pi}{B}}\frac{p_k(r_0)}{\lambda_k + s}\frac{z_k(x_1)}{r_1^{1/2}k^{1/4}(r_1)}$$
$$p(r_1, t) = \frac{1}{2\pi i}\int_{\sigma_0 - i\infty}^{\sigma_0 + i\infty} P(r_1, s) e^{st} ds$$

If the functions $k(r_m)$, $I(r_m)$ are known for N points $r = r_m$, m = 1, 2, . . ., N, then the values of the function p(r, t) are determined for all points $r_m$.

Specifically, if the function k(r) is known in the segment [r', r''], the integral I(r') is known, and the functions $\varphi(t)$ and q(t) are known, then the solution of the direct problem (1), (2) is uniquely determined in the half-band $\{r' \leq r \leq r'', t > 0\}$, and does not depend on the values of k(r), h(r) outside of the interval [r', r''], which, generally speaking, remain indeterminate.

From what has been said it follows that if for the pair of functions {k(r), h(r)}, (1), (2) and the following condition are satisfied:

$$p(r_m, t) = \varphi_m(t), \quad m = 0, 1, 2, \ldots, N \tag{13}$$

there exists an infinite set of pairs $\{k^*(r), h^*(r)\}$ for which (1), (2), and (13) are valid. For this, satisfaction of the conditions is sufficient.

The corresponding operators (8) for the pairs $\{k(r), h(r)\}$ and $\{k^*(r), h^*(r)\}$ coincide.

The following relationships are valid ($m = 0, 1, 2, \ldots, N$):

$$k(r_m) = k^*(r_m), \quad I(r_m) = I^*(r_m) \tag{14}$$

$$I(R) = I^*(R), \quad I^*(r) = \int_{r_0}^{r} \frac{dr}{\sqrt{k^*(r)}}$$

In actuality, every function k(r) satisfying (14) can be varied without breaking down relationships (14), and maintaining the value of $dk/dr|_{r=r_0}$ constant. For a changed value of $\delta k(r)$, the function $\delta h(r)$ is so chosen that the function $f(x)$ in the operator (8) will remain as before. From the form of $f(x)$, it follows that this can always be done.

The constant entering into the boundary condition does not change, since the values of $k(r_0)$, $k'(r_0)$ remain invariable.

Thus, from supplementary information of the type of (13) it is impossible to uniquely determine the functions k(r) and h(r). However, this can be done if the output and pressure of the well are measured under other operating conditions. With $r = r_0$, let the solution of Eq. (1) be known with two sets of boundary conditions [$p_0(r)$ is some unknown function]:

$$2\pi r_0 k(r_0) \frac{\partial p_1}{\partial r}\bigg|_{r=r_0} = q_1(t), \quad p_1(R) = 0 \tag{15}$$

$$p_1|_{t=0} = 0$$

$$2\pi r_0 k(r_0) \frac{\partial p_2}{\partial r}\bigg|_{r=r_0} = q_2(t), \quad p_2(R) = 0 \tag{16}$$

$$p_2|_{t=0} = p_0(r)$$

$$p_{1,2}(r_0, t) = \varphi_{1,2}(t) \tag{17}$$

We shall show that conditions (15)-(17) uniquely determine the functions k(r), h(r). Analogously to what has gone before, for the Laplace transforms $\Phi_{1,2}(s)$ we obtain

$$\Phi_1(s) = -Q_1(s) \sum_{k=1}^{\infty} \frac{p_k^2(r_0)}{\lambda_k + s} \tag{18}$$

$$\Phi_2(s) = -Q_2(s) \sum_{k=1}^{\infty} \frac{p_k^2(r_0)}{\lambda_k + s} + \sum_{k=1}^{\infty} \frac{c_k p_k(r_0)}{\lambda_k + s} \tag{19}$$

$$c_k = \int_{r_0}^{R} r p_k(r) p_0(r)\, dr$$

The second term in the right-hand part of (19) is a result of taking account of the nonnull initial data under conditions (16).

As has been shown, relationship (18) completely determines the corresponding operator (8). From (19) we obtain an expression for $c_k$,

$$c_k = p_k^{-1}(r_0) \operatorname{Res}_{\lambda=\lambda_k} G(-\lambda)$$

$$G(s) = Q_1^{-1}(s)\,(Q_2(s)\Phi_1(s) - Q_1(s)\Phi_2(s))$$

We express the coefficients $c_k$ in terms of the known normalized functions of the corresponding operator;

$$c_k = \int_{r_0}^{R} r p_k(r) p_0(r)\, dr = \sqrt{\frac{\pi}{B}} \int_0^{\pi} r(x) \frac{z_k(x)}{\sqrt{\theta(x)}} \frac{dr}{dx} p_0(r)\, dx = \sqrt{\frac{B}{\pi}} \int_0^{\pi} \sqrt{\theta(x)}\, p_0(r(x))\, z_k(x)\, dx$$

Then the function $\gamma(x)$ is known:

$$\gamma(x) = \sqrt{\frac{B}{\pi}} \sqrt{\theta(x)}\, p_0(r(x)) = \sum_{k=1}^{\infty} c_k z_k(x) \tag{20}$$

Taking into consideration that $\theta(x) = \pi y'(x)2B$, we obtain an equation for $y(x) = r^2(x)$,

$$\sqrt{\frac{1}{2} y'}\, p_0(\sqrt{y}) = \gamma(x), \quad y(0) = r_0^2 \tag{21}$$

Solving (21), from formulas (10), (11), we determine the functions k(r) and h(r).

In the presence of infiltration, the equation for the pressure assumes the form

$$\frac{\partial p}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left[rk(r)\frac{\partial p}{\partial r}\right] - h(r)p + \omega(r) \tag{22}$$

In this case, to uniquely determine the three functions, the pressure $p_3(r_0, t) = \varphi_3(t)$ must be given under some third set of conditions, for example,

$$2\pi r_0 k(r_0)\frac{\partial p_3}{\partial r}\bigg|_{r=r_0} = q_3(t) \tag{23}$$
$$p_3(R) = 0, \quad p_3|_{t=0} \equiv 0 \quad (q_1(t) \not\equiv q_3(t))$$

We then have

$$\Phi_i(s) = -Q_i(s)\sum_{k=1}^{\infty}\frac{p_k^2(r_0)}{\lambda_k + s} + \sum_{k=1}^{\infty}\frac{\omega_k p_k(r_0)}{s(\lambda_k + s)}, \quad i = 1, 3 \tag{24}$$

$$\Phi_2(s) = -Q_2(s)\sum_{k=1}^{\infty}\frac{p_k^2(r_0)}{\lambda_k + s} + \sum_{k=1}^{\infty}\frac{\omega_k p_k(r_0)}{s(\lambda_k + s)} + \sum_{k=1}^{\infty}\frac{c_k p_k(r_0)}{\lambda_k + s} \tag{25}$$

The constants $c_k$, $\omega_k$ are the coefficients of the Fourier functions $p_0(r)$, $\omega(r)$, respectively, with respect to the system $p_k(r)$.

From (24) we obtain a relationship to determine the spectral parameters of the operator (4):

$$\sum_{k=1}^{\infty}\frac{p_k^2(r_0)}{\lambda_k + s} = \frac{\Phi_1(s) - \Phi_3(s)}{Q_3(s) - Q_1(s)} \tag{26}$$

Relationship (26) determines the corresponding operator (8). Further, using the last equality of (24) and (25), we obtain

$$\sum_{k=1}^{\infty}\frac{c_k p_k(r_0)}{\lambda_k + s} = \Phi_2(s) + \Phi_1(s)\frac{Q_2(s) - Q_3(s)}{Q_3(s) - Q_1(s)} + \Phi_3(s)\frac{Q_1(s) - Q_2(s)}{Q_3(s) - Q_1(s)} \tag{27}$$

From (27) we find $c_k$ and, consequently, we determine the functions k(r) and h(r).

The coefficients $\omega_k$ are determined from (24) or (25). Then

$$\omega(r) = \sum_{k=1}^{\infty}\omega_k p_k(r)$$

It is not difficult to point to different forms of the boundary conditions, uniquely determining the functions k(r), h(r), $\omega(r)$. For example, in condition (23) we can set $p_3|_{t=0} = \alpha p_0(r)$, where $\alpha$ is some constant.

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